# The Exchequer's Guide to Revenue in the Agrarian State

Bruce Winterhalder[1] and Cedric Puleston[1]
[1]Department of Anthropology, UC Davis, Davis CA 95616 USA
[bwinterhalder@ucdavis.edu; puleston@ucdavis.edu]
Corresponding Author: Winterhalder

Version 6.1; 26 July 2013
Text Total Word Count for this Document: 11,040
8 Figures
2 Tables
Online Enhancements

Draft: Please do not cite without permission of authors

## Abstract

We adopt an imagined exchequer, functionary responsible in an early polity for securing resources from its agrarian subjects, and develop a feature-rich demographic and environmental model to explore the population ecology of agricultural production in the context of Malthusian constraints and economic exploitation. We identify and characterize a peak of surplus production prior to density dependent constraints. We then characterize the taxation potential of a population at its Malthusian equilibrium. For a fixed total level of taxation the exchequer has two options: a small population taxed at a high rate, unstable to small perturbations, or a larger population taxed at a lower rate, which is stable. In a small and growing population it is more effective to tax goods; as the population approaches its density dependent equilibrium it is more effective to tax labor. Taxation interacts with stochastic yield variation to exacerbate the magnitude of periodic famines; per capita taxation having a more severe effect than a fixed total tax. We characterize the likely persistence of early agrarian states in terms of their half-life as a function of level of taxation and degree of yield variation, and we argue that fiscal mismanagement should be among the hypotheses for polity failure.

## Introduction

The exchequer of our title is a high-level functionary in an early state made up of a ruler, an oligarchy and administrative staff, and a growing population of peasant agriculturists, the subjects of the polity and the producers of its agrarian resources. He might be standing atop a peripheral temple at Tikal, Guatemala, looking out over peasant house groups surrounded by fields of corn interspersed with patches of secondary tropical forest (Fedick and Ford 1990). He might also be standing atop a small hill on the north shore of Lake Titicaca, Peru, viewing thousands of hectares of raised fields interlaced within a reticular network of irrigation canals, the small plots planted in potato and quinoa (Erickson 1987). We could find him, or perhaps occasionally her, in similar settings in other parts of the prehistoric and ancient historic world.

We imagine him contemplating his administrative assignment: to obtain for the state from this landscape and its inhabitants the maximum quantity of resources, reliably, this year and every year to follow. He must secure the goods and labor that constitute establishment power. They will support retinues of elites and craft specialists, the construction of monuments, the holding of ritual displays and other public expressions of centralized authority, the maintenance of powerful military and religious institutions, and of course, officers whose function is to census the polity's subjects and inventory and collect its revenues.

Our exchequer habitually is anxious. He has a trusted bureaucracy but he also faces significant uncertainties. These uncertainties extend well beyond the vagaries of weather and the brittle tolerance by the peasants of their exploitation. The exchequer must assess correctly what fiscal policies will be most effective at procuring resources from the scene before him, decisions that engage him deeply in processes of demography and environment. His position, perhaps his survival and more broadly that of the state, rest on his correct assessment of the mechanisms of population ecology of his subjects, their crops and their animals.

Such functionaries almost certainly existed, and they almost certainly thought carefully about their task and choices. We believe that they would have found useful the analyses we present here, rendered as policy guidelines for administrators of the peasant agrarian state. We do not envision, however, that the significance of our results requires a rational, calculating individual agent -- an exchequer -- however useful he or she is as a rhetorical device. Rather, we expect that most early states that came upon workable or even efficient policies did so by trial and error, the emulation of successful neighbors or, simply, the accidents of good luck, as well as by calculated insight. Most likely, cultural evolutionary processes subtended the design they achieved (Henrich and McElreath 2003). There must have been many that failed to get it right and struggled or disappeared as a result (Wright 2006). Those that succeeded prospered. If we can answer questions like those faced by our imagined exchequer, we may better understand why some early states thrived and others did not, however their fiscal policies were discovered, formulated and implemented.

Chiefdoms and states are about many things, ranging from personal aggrandizement and social stratification, to public works, to compelling ideology and military power (Turchin and Gavrilets 2009), but underlying all of these is the sustainable acquisition of economically useful resources (DeMarrais, Castillo, and Earle 1996; Drennan and Peterson 2011: 65). Even the expression of power via ideology has steep economic costs as beliefs in a large and dispersed population must be inculcated, made palpable through repetitive and impressive display, material and ritual and, if they are costly to the livelihood and health of the peasant population, they may require coercive enforcement. For our exchequer, the core is economics and in early agricultural societies economics is realized largely through population and agrarian ecology.

Our questions – any of which might have occurred to a prehistoric exchequer – are these: What is the relationship between increasing population density and the potential for surplus income? Under what circumstances is it better to extract resources in the form of goods or in the form of labor, and which will have the lightest impact on a potentially restive subject population. Among such policy choices as adding territory, increasing yield through technology, adding to the workload or decreasing the consumption of the agricultural population, which have the greatest potential for enhancing revenue and in what circumstances? What are the consequences for revenue generation of fluctuating agricultural yields, and the consequences of taxation for the agrarian population experiencing the fluctuating yields?

However frequently these or similar questions occurred to state-level functionaries in the past, they are questions that seldom have occurred in the contemporary writings. While there is a very large literature on the political economy of early states (Trigger 2003; Claessen and van de Velde 2008) to the degree it considers the material dimension of economics it typically is about the collection and *distribution* of resources through social and institutional networks to ensure continued state hegemony (Smith 2004). This is important, to be sure. But much less has been written about how those resources were generated through the combination of environmental resources and agricultural labor. We know about the types of crops, minerals or other raw materials involved, estimated amounts and values, perhaps the logistic (e.g., transportation) problems of their acquisition, and the manners in which they could be processed or traded to gain value. We know little about how they were produced through mechanisms of interaction among producers, and the environment and resources they exploited. It is this subject that we take up on behalf of our exchequer.

## Population Ecology as a Model Framework

Our analysis is based on a population ecology model that links together: (1) a unisexual but age-structured population with demographic characteristics representative of a 1:1 sex ratio; (2) a set of environmental parameters that delimit the area of the environment available for agriculture, determine its yield, and translate age-specific labor investment into agricultural production measured in kilocalories; (3) a set of nutritional factors that determine the age-specific consumption requirements of this subsistence population; and, finally (4) a set of curvilinear functions that translate food availability, conceptualized as a *food ratio* (*E*, or kcals available divided by those needed to sustain fertility and mortality at baseline rates), into age-specific rates of fertility and mortality.

These rates then modify the age distribution and population size of a subsequent annual iteration of the model.

In a constant environment, model dynamics are determined primarily by the food ratio, $E$. So long as $E \geq 1$, fertility is high and mortality low, neither changes and the population grows at a constant rate set by $r_0 = b_0 - d_0$ (terms defined in Table 1). As the environment becomes saturated, land availability is constrained and competitive inefficiencies emerge. $E$ drops below 1. Decreasing per capita food production elevates mortality and depresses fertility and, as these terms converge under continued growth, density dependence leads to a stable age structure and an equilibrium at which $r = 0$. Parameters of the model are set to values realistic for human populations engaged in agricultural production (see Table 1). Full technical details of the model are available (Lee and Tuljapurkar 2008; Puleston and Tuljapurkar 2008; Lee, Puleston, and Tuljapurkar 2009); this paper builds on basic dynamic properties examined in Puleston *et al*. (n.d.), following classic articles by Lee (1986) and Wood (1998). On-line enhancements to this paper contain supporting mathematical details.

## Taxation and Surplus with Population Growth

Consider an agricultural region of 1000 ha, newly settled by a group of 20 egalitarian agriculturalists, not subject to taxation. Under our baseline parameters (Table 1) over a period of about 400 years their population will expand in a sigmoid pattern (1A). The newly settled inhabitants experience a long period of relative abundance, the *copial phase* (see Puleston, Tuljapurkar, and Winterhalder n.d.), in which production of food exceeds the minimum consumption requirements for baseline vital rates (Figure 1C). As the food ratio ($E$) declines below 1, density dependent constraints begin (at year 347) to elevate mortality and constrain fertility. This initiates a *transition phase*. Growth rapidly begins to slow. Fifty years later per capita mortality just matches per capita fertility. A zero-growth *Malthusian phase* equilibrium ensues: $\hat{r} = 0$, the food ratio $\hat{E} = 0.668$, and the subsistence population numbers $\hat{N} = 13{,}509$

Three temporal landmarks characterize this growth trajectory: (1) copial surplus food production (definition below) is at its maximum in year 293 (Figure 1E); (2) the food ratio $E$ first drops below 1 in year 347 (Figure 1C); and, (3) the population growth rate $(\delta N / \delta t)$ is maximal in year 352 (Figure 1B). Figure 1D shows that the greatest rate of decline in the food ratio $(\delta N / \delta t)$ lies between the maximum surplus and $E = 1$. Due to demographic momentum there is a slight lag between the initial impact of food limitations on vital rates ($E < 1$; yr 347) and a turn-around in population growth rates, $(\delta N / \delta t) = 0$ (yr 352; Figure 1B).

**Transient Intermediate Optimum Surplus**

We define a *copial surplus* as any point along this growth pathway at which the aggregate amount of food produced is in excess of consumption optimizing vital rates:

$$S_t = N_t(E_t - 1)j_0 \qquad (1)$$

The copial surplus at time $t$, ($S_t$), is determined by the number of producers ($N_t$) times the difference between the prevailing food ratio $E_t$ at time $t$ and the minimum ratio consistent with baseline vital rates, $E_t = 1$, times the baseline age-weighted kcal requirement per individual, $j_0$. Visual inspection of Figure 1E reveals an intermediate maximum surplus at year 293, which we designate as $S^*_{t=293}$, the (*) representing a maximum. The maximum is confirmed by noting that $N_t$ is a positive monotonic and $E_t$ a negative monotonic function, the product of which must have an intermediate peak. This a transient maximum because there is nothing in the model to suggest that population growth would be arrested at this peak.

In our baseline scenario, $S^*_{t=293}$ occurs 54 years before density dependence begins to affect the producing population ($E_t < 1$; year 347). It occurs just over a century before the population comes to equilibrium (Figure 1). Perhaps more surprising, the maximum potential for copial surplus production occurs when the population is only 25% (3382/13,509) of the size it will become at equilibrium. The capacity to produce a surplus that does not degrade vital rates is greatest when a population is quite small. The delay from settlement to this copial maximum is an inverse function of reproductive rates ($r$), a direct, log-linear function of arable land ($A_m$), and an inverse log-linear function of initial settlement size ($N_0$); further details in the Enhancements, Sections A and D.

Figure 2 (Enhancements Table D1) illustrates the effects of changing the values of area ($A_m$), yield ($Y$), population work efficiency ($w$) and population consumption ($j$) on the size of the copial surplus maximum. The vertical dashed lines isolate the combination of baseline values represented in the population trajectory of Figure 1, and from which we derived the copial surplus maximum of 6.41 x $10^6$ kcal/day. The graphic depicts the form of relationship between each of the four input variables represented on the x-axis, *read one at a time*, and the copial surplus maximum. Thus, holding the other three input variables constant at their baseline values, increasing area ($A_m$) results in a linear increase in total tax revenue. This may seem unremarkable except that $S^*$ occurs long before this area is filled and with a small fraction of the number of its eventual inhabitants. Expressed as an elasticity[i], the effect of yield ($Y$), all other inputs constant, becomes more linear and approaches 1 $\left(= wY/wY\right)$ as yield becomes large relative to consumption (Enhancements Table D1f). Increasing consumption depresses total tax revenue at the copial maximum. Finally, the elasticity of surplus in response to $w$ is positive and quite high when population work productivity is low (our baseline labor effectiveness elasticity of 0.947 occurs just about the shoulder of the curve for work effectiveness in Figure 2), but it rapidly diminishes to an asymptote as $w$ grows in magnitude. With yield and area fixed, the effectiveness of increasing surplus by expanding working hours and the age classes engaged in agrarian labor is sharply limited, even at the relatively low densities of the copial surplus maximum. We note that vital rates do not enter the approximation of the copial surplus maximum other than through their effect on the baseline age structure.

It might surprise our exchequer that a relatively small, natural-fertility population of agriculturalists -- one well below its equilibrium size and, in time, significantly short of the culmination of its growth trajectory -- has significant potential for taxation, at a relative high standard of welfare measured in terms of food availability and vital rates. An exchequer intent on hastening access to this transitory surplus would promote a high baseline reproductive rate ($r_0$). An exchequer intent on increasing its magnitude will find that greater territory and higher yield (provided it is well above baseline dietary requirements of the subsistence population) have a strong, positive and linear or near-linear effect, with no intrinsic constraints. Sharp gains are possible for labor effectiveness only if it is low to begin with; these gains quickly are exhausted. In theory, reducing consumption also elevates $S^*$, at least so long as kcal consumption is adequate for a working livelihood, but the effectiveness of manipulating this input variable also is limited.

**Impact of Taxation on the Subsistence Food Ratio**

The leadership of a growing peasant state may also want to know whether it is better to take taxes in the form of agricultural output (kcals of food) or labor, and in fixed, proportional or per capita assessments. Put baldly, as might our exchequer, what form of exploitation offers the most abundant state income with the least unhappy effect on the peasantry? To answer this question, we examine the consequences of exploitation for the food ratio, $E$. We use the concept of elasticity to capture generally the degree to which $E_t$, a measure of peasant livelihood and welfare, depends on the form and severity of taxation.

We consider five forms of taxation, described mathematically in Enhancements Table B1 and shown graphically in Figure 3. The actual levels of taxation depicted in our illustration are arbitrary and altering them will change the numerical elasticities we report. However we are interested here in the structural *pattern* of response (see Winterhalder 2002), which is independent of the magnitude of taxation provided that the burden is not so great that the population ceases to be viable. The food ratio elasticities (Enhancements Table B1) all are negative, indicating that a unit increase in taxation results in a decrease in the portion of total food production available to the agrarian population after taxation.

Fixed taxes (Figure 3; Enhancements Table B1a) which remain in place across the full sweep of population growth always are more burdensome on smaller than on larger populations. Any fixed tax that can be borne by a small population becomes insignificant to a population near its equilibrium size. At low population size the impact of a fixed tax on labor is greater than that on food, a pattern that will repeat for the per capita tax option. The magnitude of the elasticity for a fixed food tax approaches $\frac{\mathrm{T}}{YA_m F_t}$, the ratio of the food tax (T) to total production, becoming very small.

In the case of a fixed labor tax (Figure 3; Enhancements Table B1b), calculating the elasticity requires that we convert labor to land worked using the parameter, *k*. The elasticity then is the maximum amount of land that could have been cultivated with the taxed labor, $k\Lambda$, divided by the amount of land the population can cultivate with the remainder of its labor hours, $A_m F_t^{\Lambda}$. Again, a feasible tax when the population is small approaches insignificance when it is large, the elasticity falling asymptotically toward the ratio of the adjusted tax to total production, $k\Lambda / A_m F_t^{\Lambda}$. As population grows, the impacts of fixed food and labor taxes converge.

A proportional tax on food production (Figure 3; Enhancements Table B1c) has a constant elasticity, negative the ratio of the fraction of production that is taxed to that which is not, $-\frac{f}{1-f}$. As depicted in Figure 3, a 15% tax on production gives us an elasticity of - 0.18 across the full range of population growth. For comparison, the conventional sharecropping payment of 50% of production has an elasticity of - 1.0.

Per capita taxes on labor and food have quite dissimilar effects on the food ratio. When the population is small, productive land is plentiful and food is relatively easily produced in abundance. Its availability is limited primarily by labor availability. Because labor is so efficient, the marginal cost to the food ratio, $E_t$, of pulling it out of production is relatively high, resulting in the high negative elasticity associated with the per capita labor tax. By contrast, in the early stages of population growth, the impact of an increase in the per capita food tax has a much more modest negative impact on $E_t$. Labor is scarce but food is easy to produce. As the population grows the relative impact of labor and food taxes inverts. The elasticity of $E_t$ with respect to the per capita food tax ($\tau$) may be thought of as the ratio of the total food tax to total production after taxation. The total tax, $\tau$□▯, grows with population more quickly than total production due to the mounting density dependent effects of competition for increasingly scarce land. As a consequence, as a habitat is filled the elasticity of a per capita food tax rises dramatically (Figure 3; Enhancements Table B1d); the cost to the food ratio of pulling scarce food from the system is high.

An exchequer, assigned to fine-tuning taxation so to minimize its impact on the subsistence population and thus avoid needlessly provoking rebellion, would read from these equations that it is best to tax a small and growing population on a per capita basis in produce. When the population begins to fill the habitat, having put the greater part of its land into cultivation, an effective exchequer will recommend switching to a per capita tax on labor. Across the population growth trajectory, the marginal cost to subsistence food availability and peasant welfare of taxation in food increases whereas that of taxation in labor declines, the tipped hour-glass form of the relationship counseling a switch from one to the other means of exploitation.

So long as the after-tax $E \geq 1$, then whatever the impact of taxes, the population will continue to grow at its maximum rate. The peasant farmers may have to work harder than they would like to meet their obligations to the state, but in other respects their vital rates and quality of life measures (e.g., $e_0$, life expectancy) are unaffected. Tax payments induce hunger and declines in vital rates to the extent that they augment density dependent processes and push $E_t$ below 1 at a smaller population and earlier than would be the case in the absence of taxes. It is at this point that choice between the forms of taxation become burdensome to producer welfare.

## Taxation at the Malthusian Equilibrium

### Per capita tax

To this point we have examined taxing the transitory surplus production of a growing population. However, we have no reason to think the population will stabilize around that optimum. Conditions remain propitious for growth right up to the extinction of the copial surplus at $E_t = 1$. In light of this, we now consider the possibilities for taxation and its effect on producers for a population that has reached a density dependent equilibrium.

At the population's Malthusian equilibrium there is an inverse relationship between the level of taxation and the size of the population, $\hat{N}$. Per capita food consumption, $\hat{E}$, determines the point at which births balance deaths; fewer workers and their families can be fed on the reduced after-tax kcals available to them. This effect is shown in Figure 4. The straight lines represent total tax revenues as a product of the per capita tax multiplied by the population size. The curve represents total tax revenues, or total production at that population level less what the population must consume to sustain itself at the equilibrium food ratio, $\hat{E}$. The line and curve intersect at an equilibrium solution for population size. At zero per capita tax we have the full equilibrium population of 13,509. As per capita tax increases, equilibrium population shrinks. Up to a point ($N = 4507$; $\tau = 2120$ kcal/individual/day), increasing the per capita rate increases total revenue to the exchequer. After that further per capita rate increases result in declining total revenue. A per capita tax rate greater than the initial slope of the total tax revenue curve will reduce the producing population to zero. At Malthusian equilibrium, the distribution of yield between kcals diverted to the state and those provisioning the agrarian producers is non-linear and zero-sum. Section three of the supplemental materials further characterizes these relationships.

Figure 5 shows how total tax responds to key environmental and production and consumption parameters. All of these relationships are similar in form to those of the copial surplus maximum (Figure 2); y-axis values, however, are different.

**Fixed Tax**

A per capita tax on the equilibrium population has only one non-zero solution (Figure 4). Our exchequer might decide instead on a fixed total tax, independent of $N$. A line representing this form of tax does not pass through the origin except in the trivial case of no tax (see Puleston and Tuljapurkar 2008), creating the possibility of two equilibrium solutions, only one of which will be stable (Figure 6).

Compared to Figure 4, in Figure 6 we rotate the fixed tax collection to the x-axis. Total revenue (T) now is the independent variable. On the y-axis we represent the two linked factors associated with a particular total tax: equilibrium population size ($\hat{N}$) and the per capita taxation rate ($\tau$) that achieves that equilibrium. The y-axes of Figure 6 are inverse and only the per capita tax is on a linear scale. The concave left curve shows the two equilibrium combinations associated with each level of total revenue. The exchequer's assessment can be met from a high per capita tax extracted from a relatively small population or a small tax extracted from a larger population.

The dashed line combinations -- high per capita rates applied to low equilibrium populations -- are unstable in the face of small random fluctuations of population; the solid line combinations are stable. We almost surely will find our Malthusian phase population somewhere on the solid line. The dynamics work like this: At equilibrium the food ratio $\hat{E}$ has fallen such that the per capita birth rate equals the death rate ($\hat{r} = 0$); less food implies the population must shrink; more implies it can grow. Imagine the population rests on the dashed line point (a). If it randomly is reduced in size from some density-independent cause to point (b), it could form a zero-growth equilibrium only if total tax collections were to fall to the level associated with point (c). Unaware of this dynamic instability, our exchequer continues to insist on collecting the original 7 x $10^6$ kcal/day. He thereby drives *E* to a level further below replacement and population shrinks again. As this dynamic repeats the population collapses, having been drawn to the zero-population boundary at the top of the graph. By contrast, a perturbation above the dashed line implies population capacity to supply more taxes which, being unclaimed by our inflexible exchequer, provide kcals to elevate *E* above replacement, ensuring population growth. This process likewise is reinforcing, and sets the population on a course toward the solid curve combinations. The same logic applied to the solid curve gives us an equilibrium that is stable to minor population fluctuations.

The exchequer faces a trade-off: greater income entails fewer subjects who are taxed more heavily (Figure 6). The exchequer can entirely forgo income and have 13,509 subjects or seek the maximum income of 9.55 x $10^6$ kcal/day and have a more modest population of 4507 subjects. Unchanged across the full array of stable equilibrium choices is the hunger of the population, set by the equilibrium food ratio, $\hat{E} = 0.668$. A policy choice must balance revenue against the benefits of population numbers. Ironically, an exchequer on behalf of the despot may prefer a relatively high per capita tax and total income in part *because* it reduces the numbers and hence the threat of revolt by the producing subjects. This would be especially tempting if the military forces were part of the elite, maintained by high revenue in sufficient numbers to control the discontent of a small producing population.

The exchequer's margin of error in making these assessments is reduced as the fixed tax rises. Two kinds of mistakes are possible, even if the environment and hence yields and vital rates are constant: (i) the exchequer may, early in population growth, prematurely impose a fixed tax at such a high level that it rests in the unstable zone above the dashed line of Figure 6, setting the population into a spiral to extirpation; or, (ii) the exchequer might impose a tax to the right of *any* stable solution (above 9.55 x $10^6$ kcal/day). In both cases, the margin of error diminishes as the levy increases. It is easy to imagine how the many uncertainties of the exchequer's information -- empirical and conceptual -- would lead either to significant (and costly) caution or to frequent crises.

**Comparison: Copial Surplus to Malthusian Maximum Tax**

In Table 2 we compare the copial surplus at its maximum with the largest tax that could be extracted under a Malthusian equilibrium, for our baseline environmental and demographic parameters. The per capita tax rates are 1896 and 2120 kcal/person/day, respectively. The maximum transitory surplus available for taxation ($S^*$) is roughly two-thirds (67.1%) of the maximum levy ($T^*$) that can be extracted at Malthusian equilibrium. The populations themselves occur in a ratio of 0.75:1 (3382:4507), and both are small compared to a population that reaches its full equilibrium size, untaxed, of 13,509. Demographic measures of population welfare are starkly different in the two scenarios, life expectancy falling from 45 to 30 years, probability of survival to age five from 77% to 65%, and the food ratio from 1.82 to a hungry 0.668. The growth rate falls from 0.0176 to zero.

## Population and Revenue Impacts of Environmental Fluctuations

We have plagued our exchequer with uncertainty, but so far it is uncertainty only about fixed parameter values and their consequences. The reality would certainly be confounded with environmental stochasticity and thus more difficult. The random elements of drought, insect infestations and other unpredictable afflictions of agricultural production and agrarian populations -- ultimately on income levels and security -- would be major sources of anxiety. To address this, we examine properties of the per capita and fixed total tax options under the assumption of randomly fluctuating agricultural yields.

**The Impact of Yield Variance and Taxation on Population Dynamics**

Figure 7 panel A shows the randomized pattern of annual yield variance that drives our simulation. Year-to-year yields are random draws from a gamma probability density function, the habitat mean yield being $Y$ = 21,000 kcal/ha/day with a fixed coefficient of variation of 0.20. The distribution is symmetrical in this range and not significantly different from a normal distribution. There is no secular trend in the data and yields are not auto-correlated in time. Over the 600 simulated years the realized mean yield is 20,847 kcal/ha/day and the standard deviation is 4238 kcal/ha/day (CV = 0.203). The lowest yield is 9,253 kcal/ha/day (year 133); the highest is 34,283 kcal/ha/day (year 182).

Panels (B), (C) and (D) show starvation mortality in the form of elevated death rates induced by drops in yield for the no tax, fixed tax and per capita tax scenarios. The initial population in each treatment is 6500. With no tax (B), stochastically elevated per capita annual death rates are quite small (average = 0.03; CV = 0.66; all averages based on years 100 to 550), however episodes of minor mortality are common. The maximum death rate of 0.23 corresponds to the least productive year, 133. Beset by frequent low-level shortages, this population nonetheless experiences 34 years (of 451, or 7.5%) in which the food ratio $E$ is greater than 1.

Panels (C) and (D) represent a fixed tax and a per capita tax scenario, respectively. The fixed tax case is set at 60% of the maximum stable tax under deterministic conditions ( $= 0.6 \times 9.55 \times 10^6$ kcal/day; see Figure 6). The per capita scenario represents the same 60% rate ( $= 0.6 \times 2120$ kcal/person/day). Taxation significantly magnifies the impact of sharp drops of yield; it does so to a greater degree for the per capita than the fixed tax case. Taxation likewise reduces the frequency with which yield variation produces a noticeable impact on death rates, again to a greater degree for the per capita case. In the fixed tax option the average death rate is 0.03, equal to the no tax case, but the coefficient of variation rises to 1.41. The maximum of 0.54 occurs again in year 133. In the per capita scenario the average rate is also 0.03, the CV = 1.88 and the maximum rate (year 133) rises to 0.91.

In the fixed-tax case the population experiences an $E > 1$ in 207 of the 451 years of interest (46% of the time); in the per capita case $E > 1$ in 286 years of 451 (or 63% of the time). Particularly noticeable in the per capita case is the period from year 133 to 253 in which downturns of yield have no discernable impact on death rates because the population is small and recovering from the year 133 famine.

Figure 7E shows how population density responds to yield fluctuations operating through mortality. Our initial figure of 6500 places the population in the copial phase but after the peak of copial surplus (Figure 1). It is roughly half of the equilibrium population size achievable without yield variation. The no tax population grows for just over 100 years and then settles into constrained fluctuations with a mean size of 11,789 (CV = 0.06). It averages 87% of the size of its counterpart in an environment without stochastic yield variance; it is stationary (not exhibiting any long-term, directional trends). The fixed-tax case reduces the population to an average of 5008 individuals; it is locally extirpated in year 573. The per capita tax scenario population escapes extirpation, but it is severely diminished in size, averaging only 3650 individuals. The fixed-tax population averages just over 53% of its constant-environment, 60% tax equivalent population

(5008/9364). The per capita tax population is reduced to just over 54% of its analog (3650/6716).

We highlight nine observations from these panels:

(i) Spikes in the per capita death rates are synchronized among the three scenarios, but they become larger and less frequent under fixed taxes, a trend that grows with per capita taxation. The more severe the famines, the less frequent they become. Post famine, a sharply reduced population returns to the copial conditions associated with a high food ratio and thus an enhanced ability to buffer subsequent shortfalls.

(ii) Taxation sharply suppresses average population density (see also Figure 6). With stochastic yields, the no tax population averages 11,789. For a fixed tax it is 5008; in the per capita tax scenario it is 3650. What is expropriated as taxes is unavailable for subsistence.

(iii) While famine has an immediate and dramatic impact on mortality, food surfeit has only a gradual effect on population recovery. The asymmetric saw-tooth pattern evident in panel (E) is the result. Slow recovery is most evident in the per capita tax population following year 133.

(iv) Populations are vulnerable to a short sequence of closely spaced insults in which they suffer cumulative losses faster than above-average yields and thus growth can compensate for them. The fixed tax population (panel E) is extinguished by such a sequence (years 500 to 570). The average population size under stochastic conditions will be less than its size for the same *mean* conditions without fluctuations (see Lee, Puleston, and Tuljapurkar 2009 eq. 2). In our simulation the stochastic yield averages 20,846 kcal/ha/day or 99.3% of the deterministic baseline value of 21,000 kcal/ha/day. However, the stochastic no tax population (11,789) averages only 87.3% of the deterministic one (13,509). The stochastic yield, fixed-tax population (5008) is only 53% of its deterministic counterpart (9364); the per capita tax population 54% (3650 vs. 6716). These differences measure the cumulative and enduring impact of repeated downward pressures on population, which swamp opportunities for growth.

(v) The food ratio for the stochastic no tax population is modestly improved over the deterministic condition ( $\bar{E} = 0.76$ vs. $\hat{E} = 0.67$), but there are only 34 years out of 451 (7.5%) in which $E \geq 1$. The fixed tax population enjoys an average food ratio of $\bar{E}$ = 0.98, and in 46% of its years, $E \geq 1$. In the per capita tax population the corresponding numbers are $\bar{E}$ = 1.27 and 63%. It is an ironic consequence of Malthusian dynamics that a heavily taxed and thus episodically famine-prone population nonetheless eats well in one year of every two, whereas the untaxed population goes hungry nine years of ten.

(vi) On several measures, the fixed tax regime has less impact on the subsistence population than the per capita tax regime. At high population densities a fixed tax is spread over more individuals, reducing their effective individual tax rate. When the fixed tax population is small the individual's obligation rises. However, small populations also have the buffer of surplus capacity ($E_t > 1$) with which to meet the elevated claim of a fixed tax. In effect, large numbers buffer the impact of fixed taxes at high density; high surplus productive capacity buffers the impact at low density. By contrast, a per capita tax set at the same 60% rates is not diluted by large population size. When yields spike downward, mortality is disproportionately large. When populations are small, the 60% per capita rate remains at 60%, but again it is easier to absorb because surplus production is high. The near-zero impact of fluctuations on the per capita tax population for 150 years after the catastrophic famine in year 133 (panels C and E) illustrates these phenomena.

(vii) If local extirpation occurs in the per capita tax case, it will be caused solely by an environmental insult so large, or delivered to a group so vulnerable, as to drive the population to zero in a single famine event. Yields are so poor that there is nothing left after paying taxes. But so long as $N > 0$, the per capita tax population always has the capacity to recover. This is not true for the fixed tax scenario. Some combinations of tax level and population density are inherently unstable and drawn toward zero population (Figure 6). Downward spikes of yield potentially act in concert with these internal dynamics to cause population collapse.

(viii) Our two tax scenarios also play out quite differently from the exchequer's perspective. A fixed tax rate generates revenue with zero volatility (so long as the peasants survive) and a higher average return, extracted from a somewhat larger agrarian population that is less well fed. The downside is the risk of complete population collapse. Per capita taxation results in somewhat smaller average income matched to high levels of income volatility (population and income move in constant proportion), along with a smaller average agrarian population that is better fed.

(ix) Finally, we wish to note the difficulty of accurately reading the population dynamics and thus political-economy consequences we have discussed (panels B through

E) directly from yield dynamics (panel A). Yet, this kind of reading and interpretation is routinely asked of paleo-climatological and -ecological data series used to interpret subjects like the success or failure of prehistoric states. Population ecology models are necessary if we are to reliably interpret the nonlinear dynamics interposed between physical environment, yields, population and political economy.

**Population Survival under a Fixed Tax Regime**

A fixed tax gains a stable revenue stream for the exchequer so long as the agrarian population providing it remains viable. Viability however can be challenged by internal dynamics that draw the population to zero (Figure 6) acting in concert with yield fluctuations (Figure 7). By repeatedly simulating various levels of fixed tax and degrees of environmental variability (expressed as yield CV) we can examine the how these factors interact in affecting population viability. We observe that the likelihood of extirpation does not change with the length of time that a population has survived, allowing us to express the frequency of extirpation as a half-life (Figure 8).

As expected, both an increasing tax rate and an increasing yield variance reduce survival probabilities of the subsistence population. The bent-elbow form of the relationship is striking in the sharp opportunities it affords and constraints it places on our exchequer. For instance, a tax rate of 40% (of the maximum deterministic tax) offers nearly infinite population survival prospects if the yield CV is below 0.28. It offers almost immediate threat of collapse if the yield CV is above 0.38. If yield CV < 0.20, a 60% tax rate is indefinitely survivable; if yield CV > 0.50 a 20% tax rate is a serious hazard. If the long-term matters, our exchequer has little room for error. Environmental variance can sharply constrain the degree of state-level extraction consistent with state (and agrarian population) survival.

## Discussion: Financing the Peasant State

### Surplus

Surplus has a convoluted and fraught intellectual history in anthropology (Pearson 1957; Harris 1959). We have defined surplus as the difference between copial phase production and consumption requirements for baseline vital rates. We obtain a copial surplus by parameter assignments determining which age groups work and for how long. A time-minimizing (Smith 1987) or Chayanovian (1977) approach would have our agrarian producers adjusting their effort so to just maintain $E = 1$, and there would be no copial surplus. Identifying production above $E = 1$ as surplus is a conceptual matter and not an endorsement of a particular political economy.

### Copial Surplus, Malthusian Tax

If a population is sufficiently early in the copial growth phase, because it is recently settled in a new habitat, long established there but recovering from a crisis of depopulation, or is experiencing a density-dependent release due to a technological or related innovation, then it will experience a peak of potential surplus production at small numbers, well before density-dependent limitations come into play. This copial surplus maximum ($S^*$) is transient, but elevated surplus production can be prolonged (e.g., 100+ years for our baseline scenario; Figure 1A). Taxation can be extracted without having a demographic impact on the agrarian population's vital rates or quality of life variables.

The subject population enjoys abundance and well-being, although their work effort is above the level that would sustain these amenities in the absence of taxation.

The equilibrium or Malthusian extraction of taxes (T) is of a different nature. It is obtained from a population suffering the Malthusian duress of suboptimal food intake, low fertility, high mortality and a shortened life expectancy. Under such circumstances there is no surplus, as defined in the copial phase. Taxation under the conditions of Malthusian equilibrium entails a direct trade-off: revenue versus subject population density. Under our baseline conditions and focusing only on the stable outcomes (Figure 6), each per capita tax rate from 2120 kcal/individual/day down to zero determines an equilibrium population density ranging from 3875 up to 13,509 individuals and a total tax collection ranging from 9.55 x $10^6$ kcal/day down to zero. Although individual tax and population size vary across this set of revenue expectations, hunger and vital rates (fertility, mortality and life expectancy) do not. Life under Malthusian conditions is equally punishing, whatever portion of production is redirected from agrarian population mass to elite income.

Whether writers favor or are critical of theories granting population a causal role in the origins of the state, they typically conceptualize causation in terms of "population pressure." For instance, in Carneiro's (1970) model population growth in a circumscribed and over-full landscape provokes conflict, subjugation of the weaker by the stronger political units, forcing greater integration in the form of hierarchy. But population pressure is not the only circumstance in which population might be important. Surplus co-opted by authorities can be used to build powerful state-level institutions. It can be produced in abundance by a population that is well below the size that invokes population pressure and density-dependent constraints.

**Taxing Goods, Taxing Labor**

We have shown that the cost of taxation in goods (kcals) is relatively low in the early stages of population growth but jumps to a high level as the population approaches saturation. The impact of imposing taxation in the form of labor follows the reverse pattern, generating a switch-over point late in the copial phase (Figure 3). Early in a growth trajectory per capita production exceeds consumption by a wide margin; food is abundant but labor is in short supply. There is low marginal cost to the food ratio from taxing kcals but greater impact to taxing labor. Late in the growth trajectory food is dear but labor redundant. The marginal cost of taking kcals is relatively high, but the marginal cost of removing labor from a production system in which it has little to contribute is low.

We have stated these elasticities in terms of their impact on the food ratio. While the per capita labor tax has the greatest effect on $E$ at low density, it may not matter to vital rates and well being so long it leaves $E \geq 1$. The same is true when the elasticities have reversed in the Malthusian phase. The food tax has the greatest impact, but this impact is expressed through its effects on population size, not life expectancy and vital rates. Those are determined solely by Malthusian constraints. Taking taxes in kcals suppresses the population size to a greater degree than a labor tax, but it will not further diminish the food ratio. Drennan and Peterson (2011: 75) state that the magnitude of the population investment in public works "can be thought of in terms of labor, irrespective of whether the population contributes labor or goods. . ." While it is true that goods and labor can be converted one into the other using kcal equivalents, population ecology dynamics tell us that the impact of drawing on one or the other source of revenues can be quite different.

Culbert (1988 : 99) has proposed that the collapse of Classic Period Maya polities was due to an agricultural labor shortage, as agrarian workers were unwisely diverted to build monuments or serve the state in other capacities. Our model suggests this is an unlikely hypothesis. The Terminal Classic collapse capped a long period of population growth to high densities, precisely the circumstance in which the marginal agricultural return to workers is low. With much of the population redundant so far as the capacity to produce yields from agrarian labor, it safely could be diverted to state activities with little or no effect on total production.

**Options for Affecting the Magnitude of Potential State Revenue**

The amount of surplus an exchequer might expect is a function of consumption, labor productivity, habitat area, and yield. We review the exchequer's options in this respect, working from the least to potentially the most effective in generating revenue in the early phase of growth (Figures 2 & 5).

<u>Consumption ($j_0 = J\rho_0$)</u>. A unit increase in per capita consumption causes a decline in maximum copial surplus. The relationship is near linear at a rate of $-2j_0 / w_0 Y = -2 / E_m$ so long as density independent output ($w_0 Y$) is significantly greater than consumption ($j_0$) (Enhancements Table D1h). An exchequer might be interested in the extent to which surplus could be elevated by suppressing consumption. In our model

we assume demanding agricultural labor, but we do not vary consumption by work effort or model the effects on effort of suppressing consumption. Since we are measuring consumption strictly in metabolic terms, the range over which it can vary is quite constrained, making it a poor candidate for extracting enhanced surplus.

Labor productivity ($w_0 = Hk\phi_0$). The elasticity of revenue to labor investment is the same as for consumption but is positive instead of negative (Enhancements Table D1g). Increasing labor effectiveness -- achieved through raising the length of the workday for the age-class working the most ($H$), the area worked per unit time ($k$), or the commitment of a greater range of age classes to longer work-days ($H\phi_0$) -- increases surplus. This variable can have a dramatic impact on surplus only if the initial effort is quite small (Figure 2). If the exchequer is dealing with a population that works short hours, cultivating a small area, and employs a small percentage of its age distribution in agricultural work, there are significant gains to pressing for greater effort from a wider array of individuals in the population. The gains are, however, asymptotic and fairly quickly exhausted. Even labor-saving innovations as seemingly effective as the introduction of the ox-drawn plow -- which in our model would operate through $k$ -- have to be appraised in this context of rapidly diminishing returns.

Habitable agrarian landscape ($A_m$). Increasing the area in agrarian production is a sure means of increasing surplus. The elasticity of maximum copial surplus production $S^*$ to $A_m$ is 1, meaning a 10% increase in area of the same agrarian quality reliably yields a 10% increase in surplus, irrespective of scale (Enhancements Table D1e). The positive elasticity of return on annexing habitat, its constancy, and perhaps its simple observability may be one reason that many archaic states regularly were expansionist in character.

Yield ($Y$). Yield likewise has a positive and near linear effect for values of production ($w_0Y$) that are significantly greater than consumption ($j_0$), with an elasticity at our baseline values of 1.94, nearly twice that for area (Enhancements Table D1f). Yield-increasing technologies like irrigation, terracing or raised fields, manuring or improved cultigen productivity have high potential to increase surplus. Limits to these gains would be technology specific. Irrigation of drylands might double yield, but twice as much irrigation is unlikely to double it again.

Although the numerical results for our baseline dataset are different (compare Enhancements Table D1 and Table D2), the elasticity graphs for the Malthusian equilibrium tax have the same general form as those for the copial surplus (compare Figure 2 with Figure 5). The generalizations we have just cited for the copial surplus will also apply to the Malthusian equilibrium tax.

Our exchequer might through experience or analysis come to rank options as we have done. But, our model also points in the direction of a much less conscious or agent-oriented approach. All else equal, states with agrarian producers that consume relatively little and work hard will have access to greater copial surpluses and Malthusian tax than their counterparts without these advantages. However, the gains available through reduced consumption and enhanced labor productivity are modest compared to the reliability and expandability of adding arable land and the large potential of increasing its yield.

**Malthusian Fixed Tax Equilibria**

An exchequer intent on a fixed resource stream will probably be unaware that this option entails two equilibrium points for each feasible level of state income (Figure 6). A high tax rate applied to a small population is an unstable option, which will either collapse to extinction through failure of the agrarian population to replace itself or move to the second and stable equilibrium combining a large population with a lower effective tax rate. There also is a tax limit above which no agrarian population can persist. The form of these relationships means that the exchequer who seeks a fixed but increasing revenue stream courts an ever greater risk of misjudging and crossing the boundary from persistent tax and population combinations to those that are doomed from endogenous dynamics.

**Effect of Stochastic Variation on Revenue**

Interactions between stochastic yields and taxation complicate the exchequer's job. In a fluctuating environment taxes exacerbate mortality from famine and they suppress average population size. Higher levels of taxation or imposition of taxes with greater impact (e.g., per capita taxation) increase the severity of periodic famines but reduce the frequency of food shortfalls. Consequently they lead to higher *average* standards of population welfare. This is not because taxation is good, but because taxation combines with yield variance to set back growth via occasional famines, lessening the unhappy consequences of Malthusian density dependence during inter-famine periods. Over-taxation or increasing yield variation enhance the likelihood of collapse by imposing sharp boundaries on the half-life, thus likelihood, of state persistence.

**The Archaeology of State Revenue**

Effective prehistoric tax rates can be calculated from data on the labor and materials requirements of public works and the size and consumption of non-producing political elites (Drennan and Peterson 2011: 75-76). Political centralization and the relative size of the non-producing population may be archaeologically visible by assessing the extent of public state-level architecture relative to rural settlement and agrarian habit. For instance, Steponaitis (1981) has shown for the Middle to Terminal Formative Period in the Valley of Mexico that data representing settlement size (a proxy for population) and the catchment area of the surrounding agricultural habitat (representing potential food production) can be used to calculate measures of political centralization and the relative amount of food which state functionaries and elites at higher level sites must have mobilized from the productive population at lower levels in the settlement hierarchy. Transfers of tribute from the egalitarian nucleated village level to local centers, and from local to regional centers both suggest that about 16% of production was directed to maintenance of the political establishment. For comparison, our 60% tax rate (assessed as a percentage of the maximum sustainable fixed tax) converts to a 28.5% rate if applied to total production, the metric used by Steponaitis.

**Fiscal Mismanagement and State Decline**

In an earlier application of this model (Puleston, Tuljapurkar, and Winterhalder n.d.), we noted that the transition from a copial growth phase to Malthusian density dependence can be abrupt. It typically is more rapid and severe the more benign the circumstances that preceded it. We suggested that the abruptness of Malthusian constraints should be considered among the strains that could lead to state collapse (Drennan and Peterson 2011; Turchin and Gavrilets 2009; Trigger 2003). Analyses presented in the current paper add fiscal mismanagement to the list of hypotheses. We began by assigning our fictional exchequer an anxious mien; our reasons for doing so should now be apparent. Selecting the form of revenue to extract from an agrarian population, setting the level of taxation, and then adjusting it to balance demands for state income with the environmental and demographic exigencies that can threaten the producing populations are complex problems, fraught with high levels of uncertainty and risk and, the occasional surprise. We do not know of empirical work by which we might assess the fiscal hypothesis, but the ability of archaeologists to indirectly estimate taxation suggests that it is feasible.

## Conclusion

Research on the political economy of the state has focused almost entirely on resource distribution and the means by which state power is consolidated and exercised through elite manipulation of resources. The production of those resources is seldom examined and in this respect the extant literature is seriously incomplete. Even a basic examination such as we have given of the population ecology of agrarian producers subject to taxation reveals mechanisms and dynamics essential to analyzing the origins, persistence and eventual decay or collapse of centralized political economies.

## Acknowledgments

This work is supported by the NSF (Division of Behavioral and Cognitive Sciences) Human Systems Dynamics, Collaborative Research proposal, *Development and Resilience of Complex Socioeconomic Systems: A theoretical Model and Case Study from the Maya Lowlands* [Proposal# 0827275]. The authors would like to thank Sheryl Gerety, Michael Graves, Patrick Kirch, Thegn Ladefoged, and Shripad Tuljapurkar for comments that significantly improved this manuscript. The Santa Fe Institute, the National Institute for Mathematical and Biological Synthesis (NIMBioS; U of Tennessee - Knoxville) and Peet's Coffee provided settings in which discussion of the results substantially improved them.

## Bibliography

Carneiro, Robert L. 1970. A theory of the origin of the state. *Science* 169 (3947):733-738.

Chayanov, A. V. 1977. On the theory of non-capitalist economic systems. In *Peasant Livelihood: Studies in Economic Anthropology and Cultural Ecology*, edited by R. H. Halperin and J. Dow. New York: St. Martin's Press.

Claessen, Henri J. M., and Pieter van de Velde, eds. 2008. *Early State Economics*. Edited by M. Aronoff. 2nd ed, *Political and Legal Anthropology Series, Volume 8*. New Brunswick, USA: Transaction Publishers.

Culbert, T. Patrick. 1988. Collapse of Classic Maya civilization. In *The Collapse of Ancient States and Civilizations*, edited by N. Yoffee and G. L. Cowgill. Tucson, AZ: University of Arizona Press.

DeMarrais, Elizabeth, Luis Jaime Castillo, and Timothy Earle. 1996. Ideology, materialization, and power strategies. *Current Anthropology* 37 (1):15-31.

Drennan, Robert D., and Christian E. Peterson. 2011. Challenges for comparative study of early complex societies. In *The Comparative Archaeology of Complex Societies*, edited by M. E. Smith. Cambridge: Cambridge University Press.

Erickson, Clark L. 1987. The dating of raised-field agriculture in the Lake Titicaca Basin, Peru. In *Pre-Hispanic Agricultural Fields in the Andean Region*, edited by W. M. Denevan, K. Mathewson and G. Knapp. Oxford: BAR International Series 359(i).

Fedick, Scott L., and Anabel Ford. 1990. The prehistoric agricultural landscape of the central Maya lowlands: An examination of local variability in a regional context. *World Archaeology* 22 (1):18-33.

Harris, Marvin. 1959. The economy has no surplus? *American Anthropologist* 61 (2):185-199.

Henrich, Joseph, and Richard McElreath. 2003. The evolution of cultural evolution. *Evolutionary Anthropology* 12 (3):123-135.

Lee, Charlotte T., Cedric O. Puleston, and Shripad Tuljapurkar. 2009. Population and prehistory III: Food-dependent demography in variable environments. *Theoretical Population Biology* 76 (3):179-188.

Lee, Charlotte T., and Shripad Tuljapurkar. 2008. Population and prehistory I: Food-dependent population growth in constant environments. *Theoretical Population Biology* 73:473-482.

Lee, Ronald Demos. 1986. Malthus and Boserup: A dynamic synthesis. In *The State of Population Theory*, edited by D. Coleman and R. Schofield. New York: Basil Blackwell.

Pearson, Harry W. 1957. The economy has no surplus: Critique of a theory of development. In *Trade and Market in the Early Empires: Economies in History and Theory*, edited by K. Polanyi, C. M. Arensberg and H. W. Pearson. New York: The Free Press.

Puleston, Cedric O., and Shripad Tuljapurkar. 2008. Population and prehistory II: Space-limited human populations in constant environments. *Theoretical Population Biology* 74:147-160.

Puleston, Cedric O., Shripad Tuljapurkar, and Bruce Winterhalder. n.d. Abrupt imposition of Malthusian equilibrium in a natural fertility, agrarian society. Davis, California.

Smith, Eric Alden. 1987. On fitness maximization, limited needs, and hunter-gatherer time allocation. *Ethology and Sociobiology* 8:73-85.

Smith, M. E. 2004. The archaeology of ancient state economies. *Annual Review of Anthropology* 33:73-102.

Steponaitis, Vincas P. 1981. Settlement hierarchies and political complexity in nonmarket societies: The Formative Period of the Valley of Mexico. *American Anthropologist* 83 (2):320-363.

Trigger, Bruce G. 2003. *Understanding Early Civilizations: A Comparative Study*. Cambridge: Cambridge University Press.

Turchin, Peter, and Sergey Gavrilets. 2009. Evolution of complex hierarchical societies. *Social Evolution and History* 8 (2):167-198.

Winterhalder, Bruce. 2002. Models. In *Darwin and Archaeology: A Handbook of Key Concepts*, edited by J. P. Hart and J. E. Terrell. Westport, Connecticut: Bergin & Garvey.

Wood, James W. 1998. A theory of preindustrial population dynamics: Demography, economy, and well-being in Malthusian systems. *Current Anthropology* 39 (1):99-135.

Wright, Henry T. 2006. Early state dynamics as political experiment. *Journal of Anthropological Research* 62 (3):305-319.

## Figure Captions

Figure 1. Surplus as a Function of Population Growth. From top to bottom the panels represent: (A) Population size, $N$; (B) Population growth rate, $\delta N/\delta t$; (C) Food ratio, $E$; (D) Rate of change in $E$ as a function of time, $\delta E/\delta t$; and (E) Surplus, measured as $S_t = N_t(E_t - 1)j_0$. $N_{t=0} = 20$. The food ratio $E_t$ has declined to 1 by the year 347, at a population size of 8734. The population growth rate, $\delta N/\delta t$, peaks at 157/yr shortly after (yr 352). The equilibrium population size is ($\hat{N} = 13{,}509$). $E_{t=0}$ is 2.99 and at equilibrium has fallen to $\hat{E} = 0.668$; $E$ equals 1.82 at the maximum surplus. $\delta E/\delta t$ reaches a minimum at – 0.0157. The copial maximum surplus $S^*$ is at year 293; it equals $6.41 \times 10^6$ kcal/day and occurs at population size $N = 3382$. At equilibrium ($\hat{E} = 0.668$), total production falls $1.03 \times 10^7$ kcal/day short of the population food requirement to achieve caloric adequacy ($E = 1$).

Figure 2. Copial Surplus Magnitude, as a Conditional Function of Area, Yield, Work Effectiveness, and Consumption. This illustration should be read as follows: Conditional on the remaining three input parameters being held at their baseline value, the total surplus available at the copial maximum responds to the fourth parameter as shown by its curve and x-axis values. For example, with consumption, area and yield at their baseline values, increasing work effectiveness in the range of 0.2 to 0.5 ha/person adds rapidly to total tax, but this increase approaches an asymptote above values of 1.0 ha/person. Each input parameter can be read individually in this manner; the illustration cannot be used to represent the joint effects of parameters. Baseline values as follows: Area ($A_m$) = 1000 ha; Yield ($Y$) = 21,000 kcal/ha/day; Consumption ($j_0$) = 2318 kcal/person/day; Work Effectiveness ($w_0$) = 0.3483 ha/person. See also Enhancements Table D1.

Figure 3. Elasticity of the Food Ratio (*E*) as a Function of Taxation. The curves in this illustration represent the relative decline in the food ratio, *E*, due to a unit increase in taxation. Early in the growth pathway of a population filling a habitat, an increase in per capital taxation of labor has a much larger impact on *E* than per capita taxation of food production; this effect reverses as the population grows toward equilibrium. The elasticities are negative because a unit increase in taxation always causes the food ratio to decline. The tax rate, and initial $t = 1$ and final $t = 500$ elasticities for each form of tax, are as follows:

| Tax type | Tax amount | $t = 1$ | $t = 500$ |
|---|---|---|---|
| Fixed food | 20,000 kcal total/day | - 0.172 | - 0.001 |
| Fixed labor | 20 hr total/day | - 0.292 | - 0.002 |
| Proportional food | 15% | - 0.177 | - 0.177 |
| Per capita food | 400 kcal/worker/day | - 0.063 | - 0.351 |
| Per capita labor | 2.5 hr/worker/day | - 0.333 | - 0.040 |

The per capita labor and per capita food tax curves cross at year 324 for this particular parameter set, 23 years before the first impact of density dependent constraints on vital rates (see Figure 1). See also Enhancements Table B1.

Figure 4. Total Tax Revenue and Population Size at Equilibrium, as a Function of Tax Rate. The solid, concave downward curve shows all possibilities of total tax revenue as function of population. It is determined by total production less equilibrium food requirements ($\hat{E} = 0.668$) at a particular population size. The straight lines show total revenue as a function of per capita tax rate and population. The intersection of the possibility and rate curves establishes the population size and total income associated with a particular rate of taxation at equilibrium. Zero tax produces our default equilibrium density of $\hat{N} = 13{,}509$ and no net revenue. Increasing tax rates lower equilibrium population but increase revenue, up to a point. The revenue maximum ($T^*$), marked by an asterisk (*), is $9.55 \times 10^6$ kcal/day, set by a per capita tax rate of $\tau = 2120$ kcal/person/day, assessed on a population ($\hat{N}_{T^*}$) of 4507 people, the equilibrium number at this rate of taxation.

Figure 5. Population Equilibrium Maximum Taxation, as a Function of Area, Yield, Work Effectiveness, and Consumption. This illustration should be interpreted by the same conventions as for Figure 2; the baseline values are the same. Note that the y-axis in this case is an order of magnitude larger ($10^7$ versus $10^6$) than that for Figure 2. See also Enhancements Table D2.

Figure 6. Stable and Unstable Equilibria of Fixed Tax Rate x Population Size, as a Function of Total Tax Collected. Each quantity of fixed-tax collection can be generated by a small population ( $\hat{N}$ ) being assessed at a high rate, or by a larger population assessed at a lower rate. The former (dashed) line is an unstable equilibrium; the latter (solid line) is stable. Unstable equilibria, if perturbed below ( $\hat{N}$ ), decline in size to extinction; if perturbed above ( $\hat{N}$ ), they move to a corresponding stable equilibrium. The arrows show the direction of these tendencies. The maximum sustainable fixed tax is 9.55 x $10^6$ kcal/day, and corresponds to a population of 4507 assessed at a rate of 2120 kcal/ person/day. Note that the y axis scales are inverse to one another, and that the per capita tax rate scale is linear while the equilibrium population size scale ( $\hat{N}$ ) is not. Further details in the text.

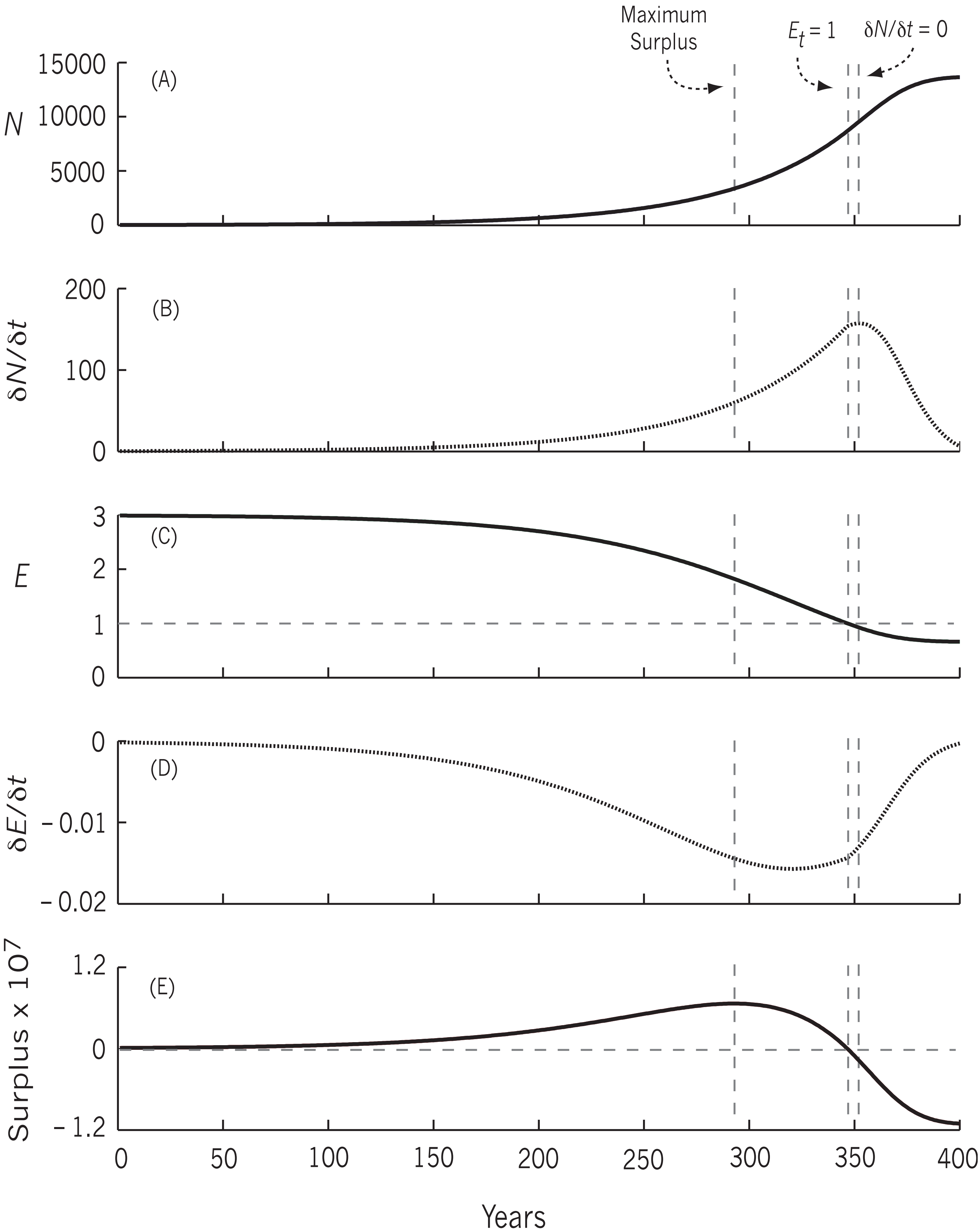
Figure 7. Mortality and Population Trajectories as a Function of Yield Fluctuations Under Three Taxation Scenarios. (A) Yields represented as a random draw from a symmetrical gamma distribution around the mean of 21,000 kca/ha/day, with a coefficient of variation of 0.2. Total mortality in the absence of taxation (B); with a fixed tax set at $5.73 \times 10^6$ , or 60% of the deterministic maximum of $9.55 \times 10^6$ kcal per day (C); the same tax rate imposed as a per capita tax (D). Panel (E) shows the population trajectories in each of the three scenarios -- no tax, fixed tax, per capita tax -- from a starting point of 6500 individuals and a representative age distribution. Note that the fixed-tax population goes extinct in year 573 of the simulation. Further interpretation in the text.

Figure 8. The Half Life (to Extinction) of Agrarian Populations, as a Function of Yield Variation and Rate of Taxation. The closed circles (20%), open diamonds (40%) and open circles (60%) represent rates of fixed taxation as a fraction of the maximum deterministic fixed rate (see Figure 6); the solid lines are a best fit polynomial. Yields are represented by random draws from a symmetrical gamma distribution, with a coefficient of variation as shown. Increasing the yield coefficient of variation for a particular tax rate shortens the expected persistence of a population; raising the tax rate for a particular coefficient of variation has the same effect. The angular or "elbow" shape of the curves implies that small changes in the degree of yield variation or taxation can dramatically shorten or lengthen population half-lives (e.g., decrease or elevate year-to-year odds of extirpation).

---

[i] Elasticity here represents the instantaneous effect on the dependent variable (copial surplus) of a unit change in the independent variable (*w*). See Enhancements Sect. D for discussion.

Figure 1



Maximum Surplus
$E_t = 1$
$\delta N/\delta t = 0$
(A)
$N$
15000
10000
5000
0
(B)
$\delta N/\delta t$
200
100
0
(C)
$E$
3
2
1
0
(D)
$\delta E/\delta t$
0
−0.01
−0.02
(E)
Surplus × $10^7$
1.2
0
−1.2
0
50
100
150
200
250
300
350
400
Years

Figure 2

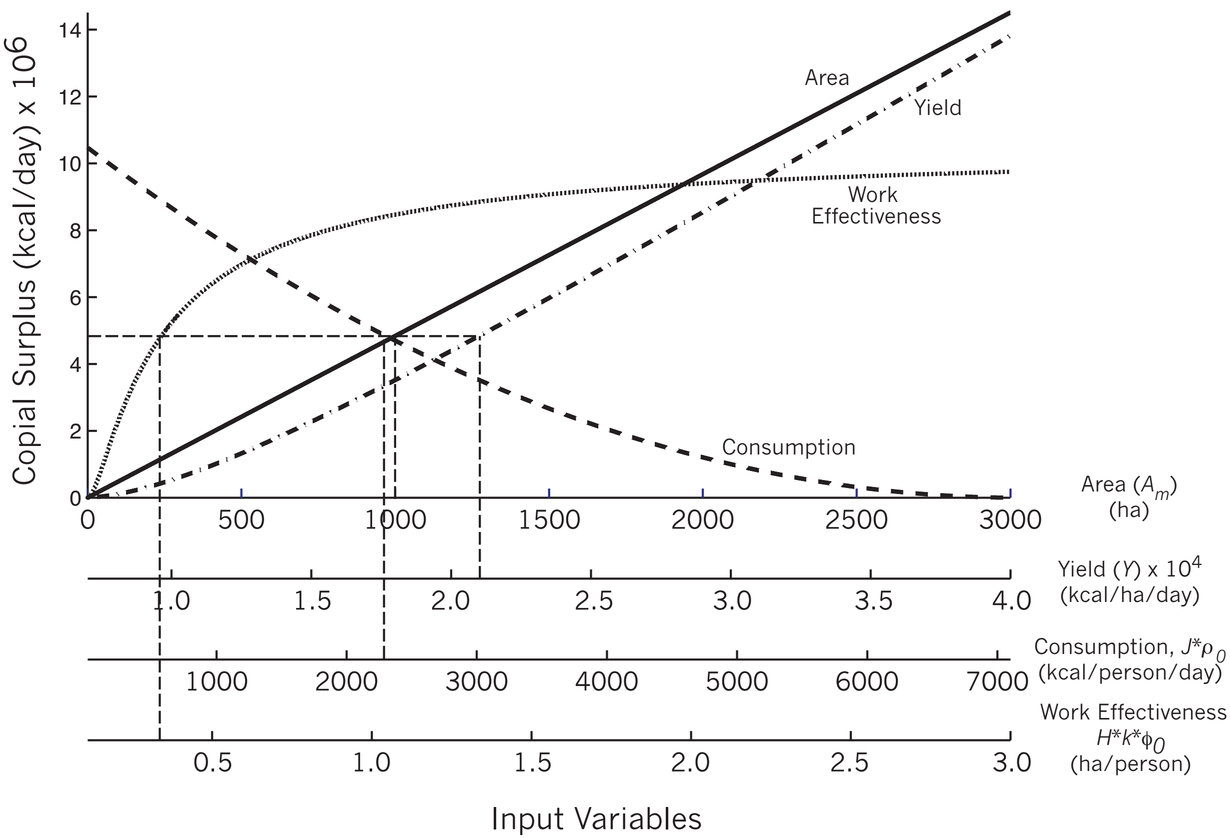
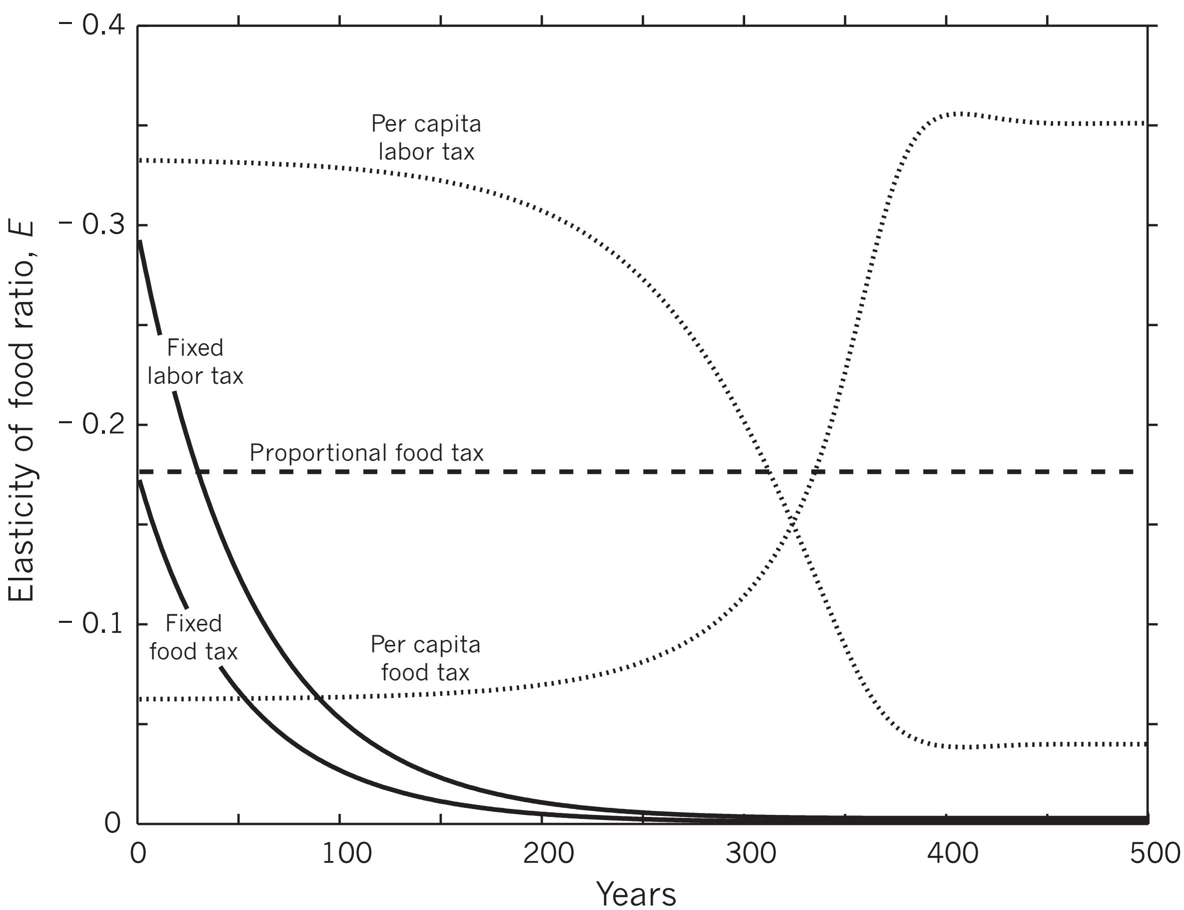
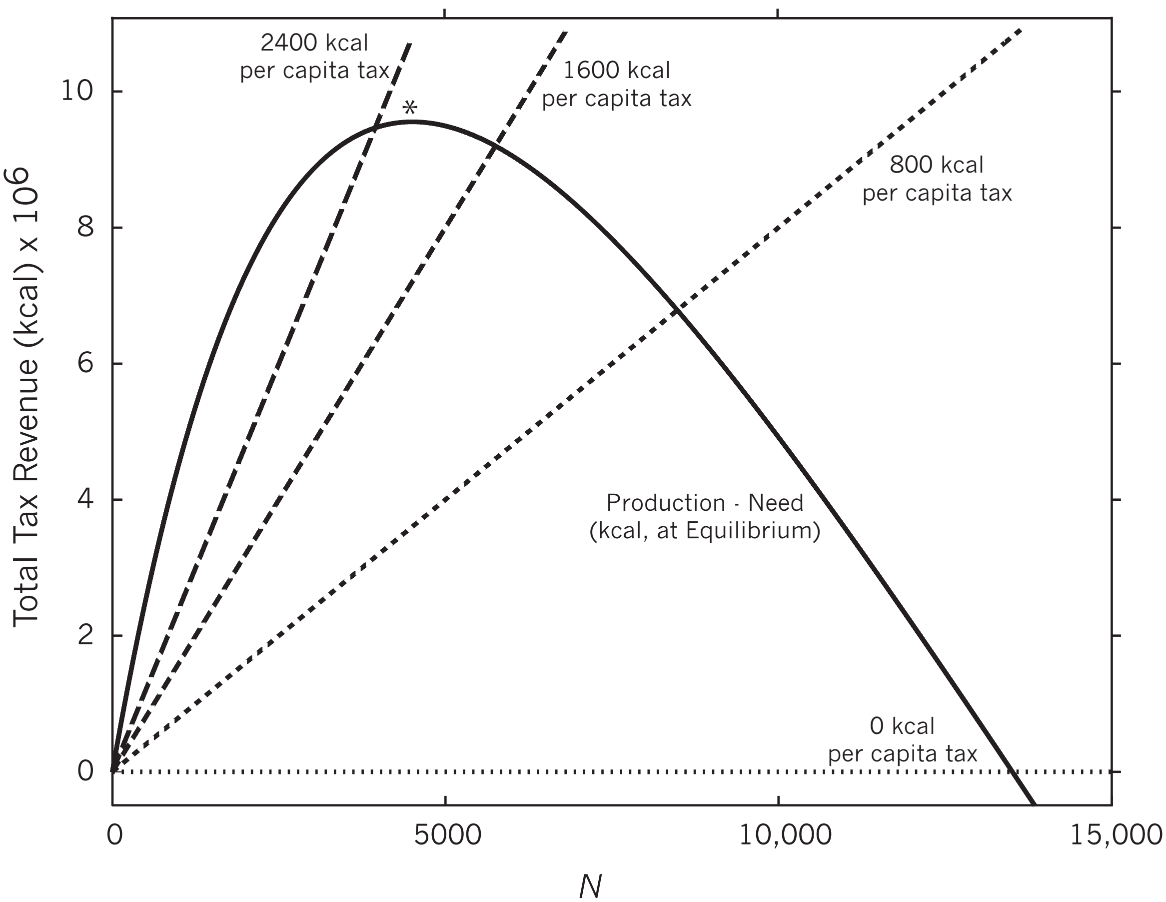

Copial Surplus (kcal/day) x 106
14
12
10
8
6
4
2
0
Area
Yield
Work
Effectiveness
Consumption
0
500
1000
1500
2000
2500
3000
Area ($A_m$)
(ha)
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Yield ($Y$) x $10^4$
(kcal/ha/day)
1000
2000
3000
4000
5000
6000
7000
Consumption, $J^*\rho_0$
(kcal/person/day)
0.5
1.0
1.5
2.0
2.5
3.0
Work Effectiveness
$H^*k^*\phi_0$
(ha/person)
Input Variables

Figure 3



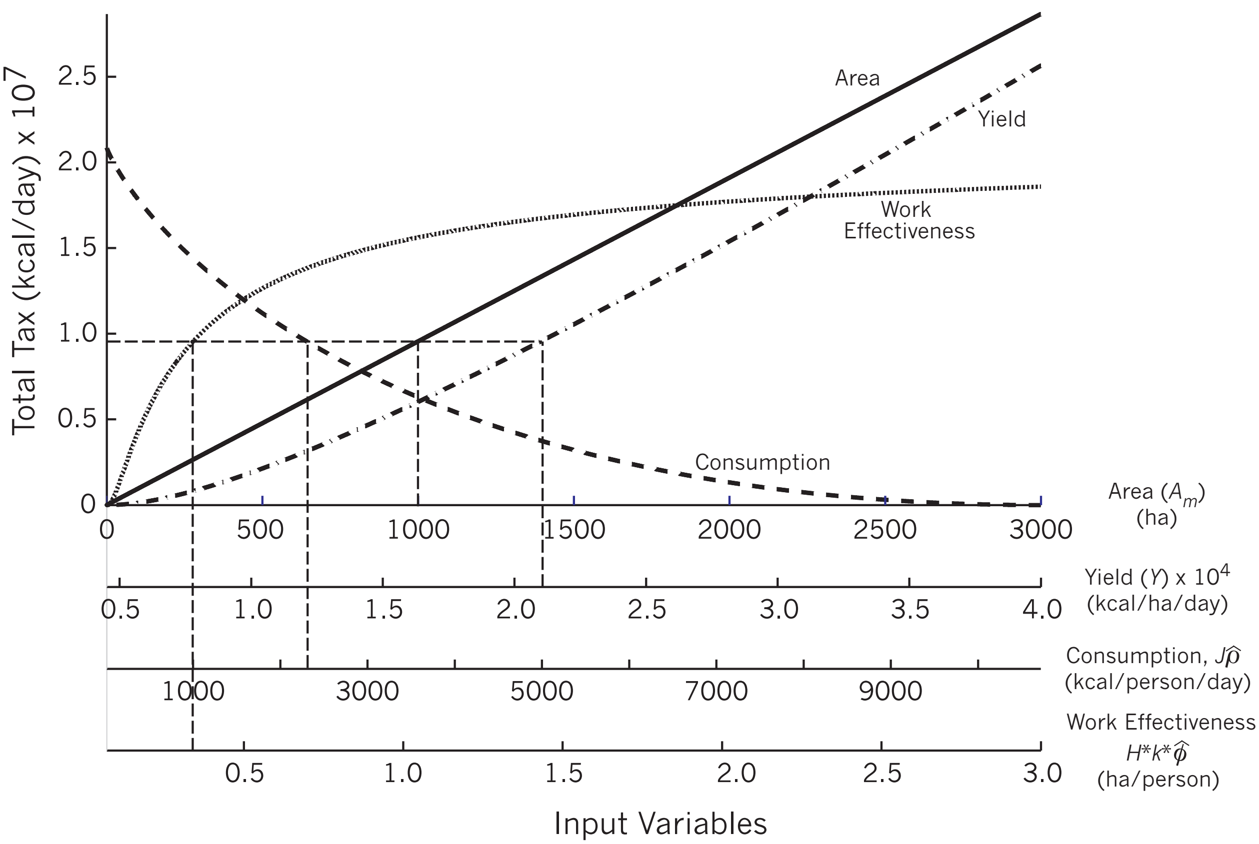
Per capita labor tax
Fixed labor tax
Proportional food tax
Fixed food tax
Per capita food tax
0.4
0.3
0.2
0.1
0
Elasticity of food ratio, E
0
100
200
300
400
500
Years

Figure 4



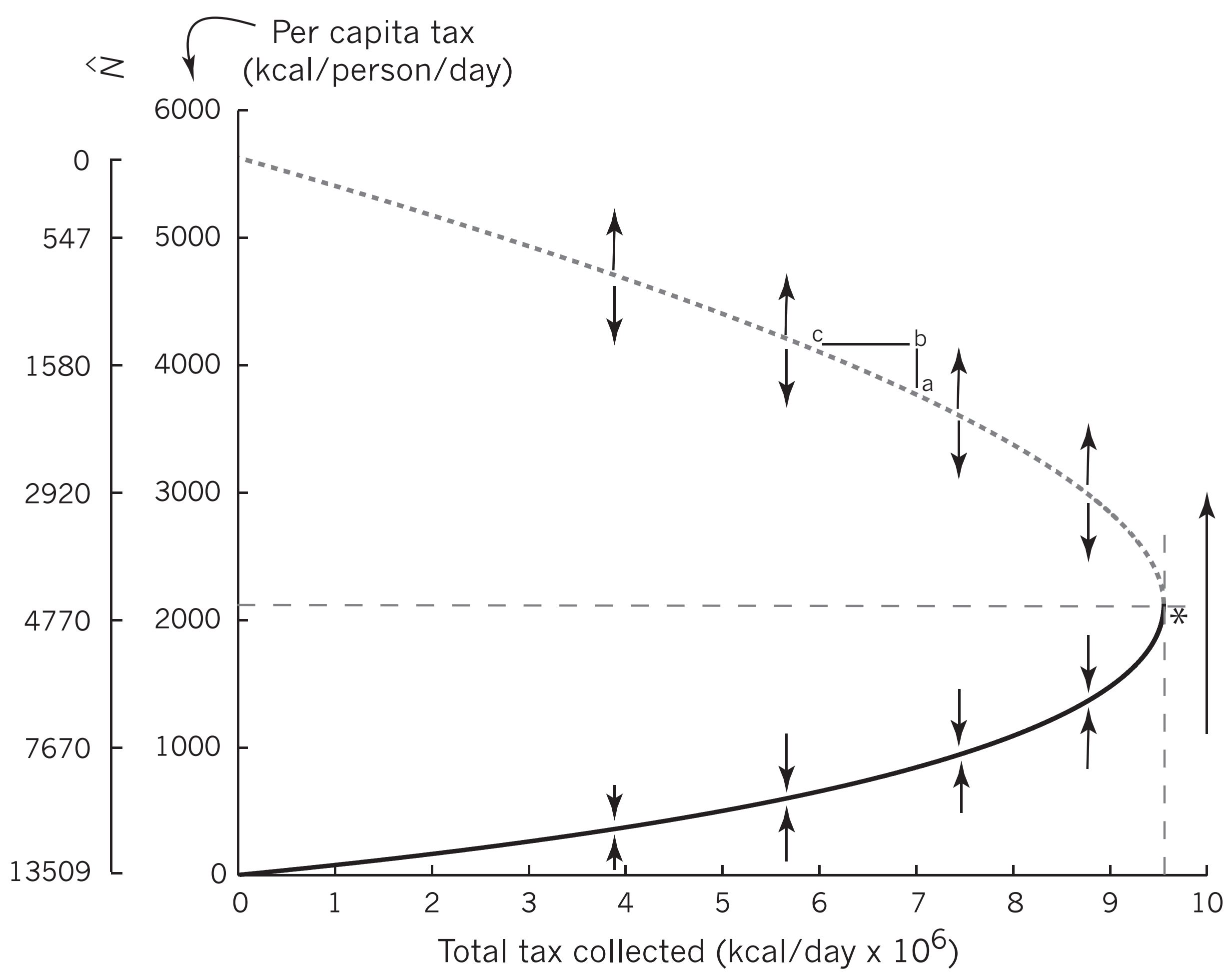
2400 kcal
per capita tax
1600 kcal
per capita tax
800 kcal
per capita tax
Production - Need
(kcal, at Equilibrium)
0 kcal
per capita tax
*
Total Tax Revenue (kcal) x 10^6
10
8
6
4
2
0
0
5000
10,000
15,000
N

Figure 5



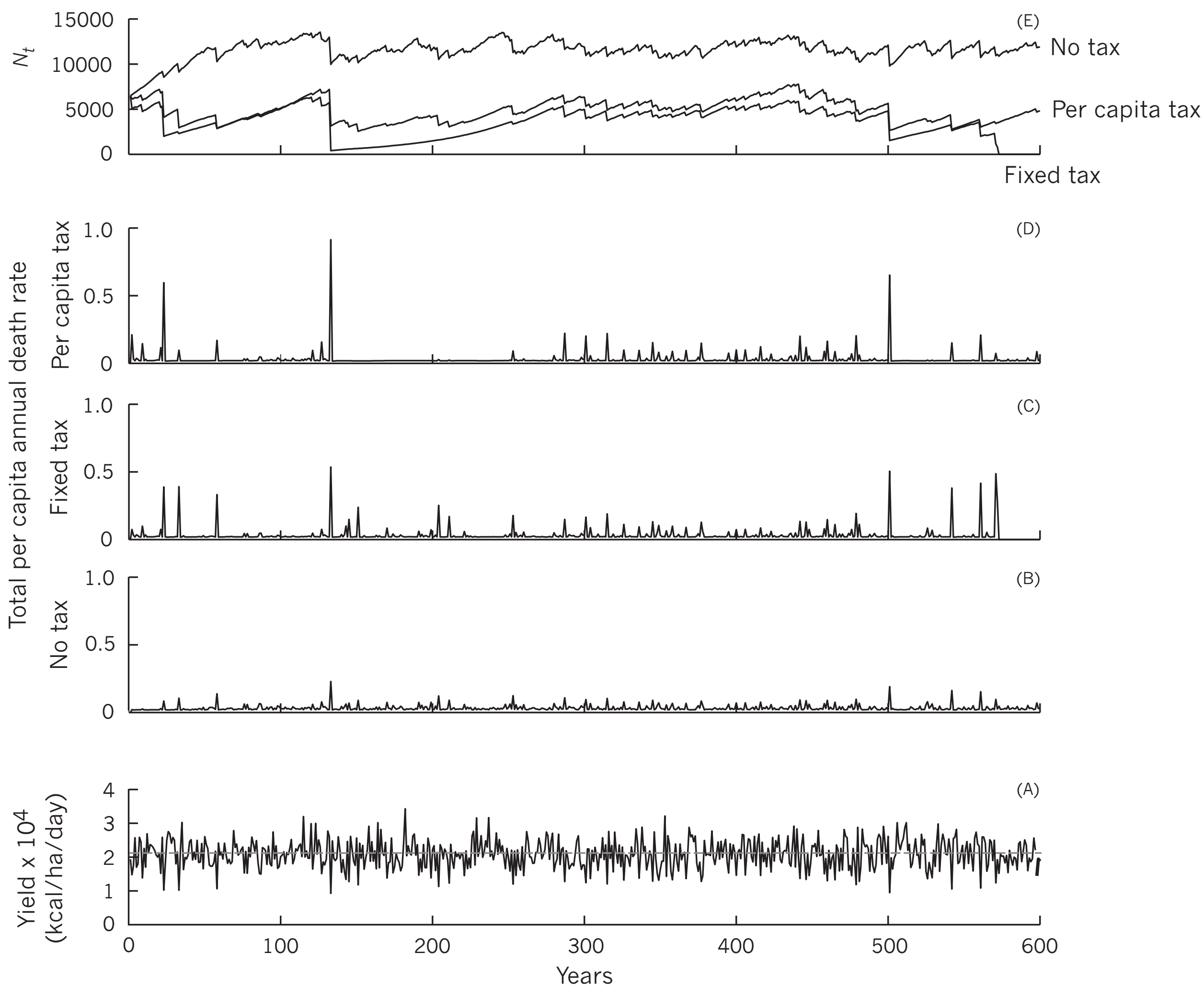
Total Tax (kcal/day) x 10^7
2.5
2.0
1.5
1.0
0.5
0
Area
Yield
Work
Effectiveness
Consumption
0
500
1000
1500
2000
2500
3000
Area ($A_m$)
(ha)
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Yield ($Y$) x $10^4$
(kcal/ha/day)
1000
3000
5000
7000
9000
Consumption, $J\hat{\rho}$
(kcal/person/day)
0.5
1.0
1.5
2.0
2.5
3.0
Work Effectiveness
$H^*k^*\hat{\phi}$
(ha/person)
Input Variables

Figure 6



Per capita tax
(kcal/person/day)
N̂
6000
5000
4000
3000
2000
1000
0
0
547
1580
2920
4770
7670
13509
c
b
a
*
0
1
2
3
4
5
6
7
8
9
10
Total tax collected (kcal/day x 10^6)

Figure 7



(E)
15000
10000
5000
0
$N_t$
No tax
Per capita tax
Fixed tax
(D)
1.0
0.5
0
Per capita tax
Total per capita annual death rate
(C)
Fixed tax
(B)
No tax
(A)
4
3
2
1
0
Yield x $10^4$ (kcal/ha/day)
0
100
200
300
400
500
600
Years

Figure 8

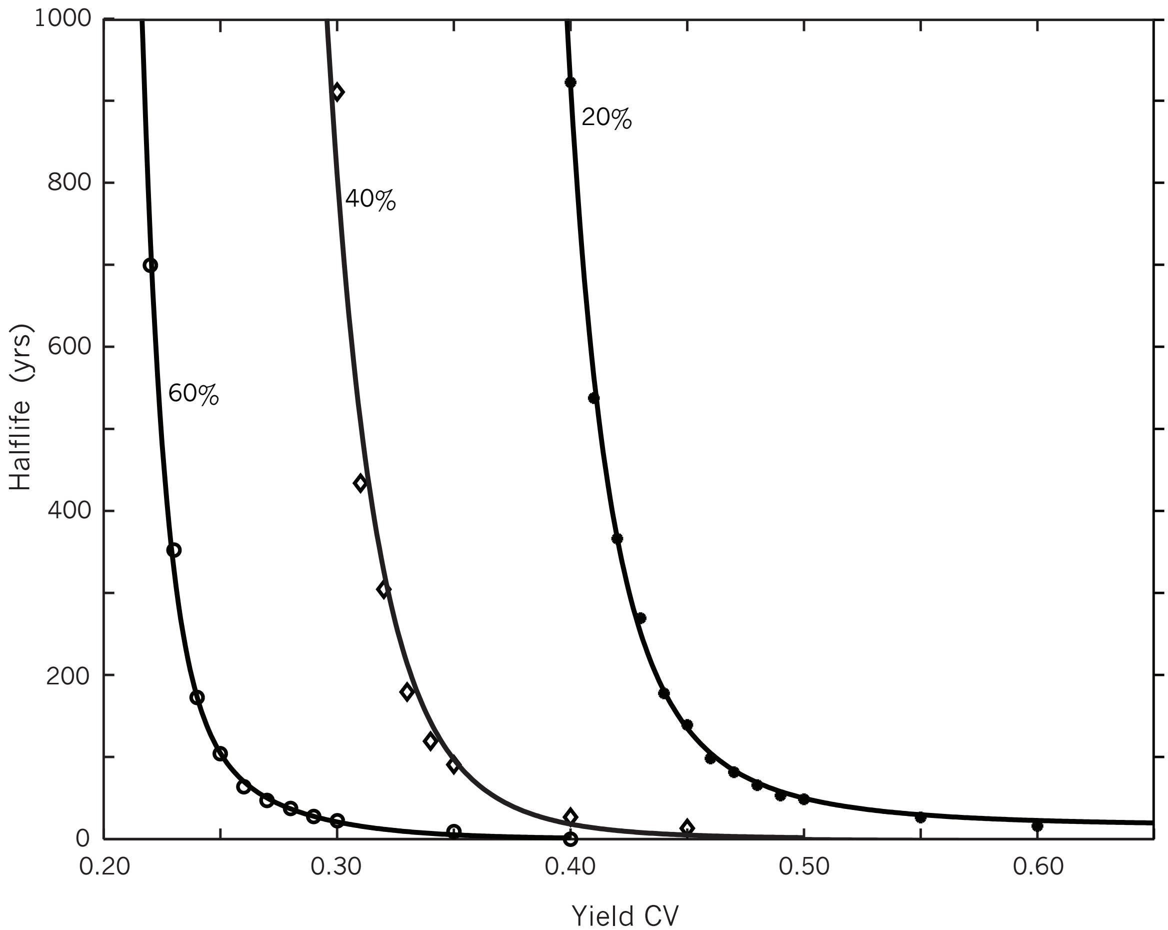

1000
800
600
400
200
0
Halflife (yrs)
0.20
0.30
0.40
0.50
0.60
Yield CV
20%
40%
60%

Table 1
Click here to download Table: Exchequer Final Table 1.doc

Table 1. Model Parameters, Mathematical Symbols and Conventions.

| Parameter | Interpretation | Value & Unit |
|---|---|---|
| $Y$ | yield/area | 21,000 kcal/ha/day |
| $H$ | longest age-specific agricultural work day | 5 hr/day/ind (or 10 hr/day/worker) |
| $k$ | area worked/hr | 0.0944 ha-day/hr |
| $\phi_x$ | proportion of hours, $H$, worked by age $x$ | $0 \leq \phi_x \leq 1$ |
| $\phi_0$, $\hat{\phi}$ | average age structure weighted hours; effective workers/person given copial or equilibrium structure, respectively | $\phi_0 = 0.738$; $\hat{\phi} = 0.723$ |
| $m_{x(0)}$ | baseline, age-specific fertility, $E \geq 1$ | daughters/woman from age $x$ to $x + 1$ |
| $p_{x(0)}$ | baseline, age-specific survival, $E \geq 1$ | probability of survival from age $x$ to $x + 1$ |
| $J$ | baseline, age-specific kcal requirement for most active age class | 2,785 kcal/day |
| $\rho_x$ | Proportion of consumption, $J$, by age | $0 \leq \rho_x \leq 1$ |
| $\rho_0$, $\hat{\rho}$ | Average, age structure weighted consumption | $\rho_0 = 0.832$; $\hat{\rho} = 0.827$ |
| $A_m$ | Arable land available | 1,000 ha |
| $F$ | Fraction of $A_m$ in cultivation | $0 \leq F \leq 1$ |
| $\alpha_x$ | Elasticity at $E = 1$, at age $x$, for survival rate, $p_x$ | $0.00279 \leq \alpha_x \leq 0.156$; $\alpha_{25} = 0.00464$ at $E = 1$ |
| $\gamma$ | Elasticity at $E = 1$, at any age for fertility, $m_x$ | $\gamma = 0.135$ at $E = 1$ |
| $w$ | $w = Hk\phi_0$, labor effectiveness, average ha/person; $w_0$ = copial phase, $\hat{w}$ = equilibrium phase | $\hat{w} = 0.3413$ ha/person $w_0 = 0.3483$ ha/person |

| | | |
|---|---|---|
| $j$ | $j$ = consumption, average weighted per age structure; $j_0$ = copial, stable age structure; $\hat{j}$ = equilibrium stable age structure | $\hat{j}$ = 2303 kcal/day<br>$j_0$ = 2318 kcal/day |
| $E$ | Food ratio | $E_{t=0} = 2.99$; $\hat{E} = 0.6683$ |
| $E_m$, $\hat{E}_m$ | Food ratio for an infinitesimally small population with either stable copial or equilibrium age structure, respectively | $\frac{YHk\phi_0}{J\rho_0}$=3.16, $\frac{YHk\hat{\phi}}{J\hat{\rho}}$=3.11 |
| $B$ | Fraction of $A_m$ the initial population could have cultivated in the absence of density dependence | $\frac{Hk\phi_0 N_0}{A_m} = 0.0070$ |
| $r$, $b$, $d$ | per capita reproductive ($r$), birth ($b$) and death ($d$) rate | $r_0 = 0.0176$; $b_0 = 0.0369$; $d_0 = 0.0192$ |
| $\hat{r}, \hat{b}, \hat{d}$ | equilibrium growth rate, births per capita and deaths per capita, at $\hat{E}$ | $\hat{r} = 0$; $\hat{b} = 0.0328$; $\hat{d} = 0.0328$ |
| $S$, T | $S$ is copial phase surplus production; T is Malthusian phase taxation | |
| ^, *, - | Equilibrium, maximum, & mean value | |
| ≈ | Approximately equal | |
| | The subscript 0 signifies a baseline value | |

Table 2
Click here to download Table: Exchequer Final Table 2.doc

$\hat{E}$ ).

| Parameter | Transitional (Copial) Maximum | Equilibrium Maximum | Units |
|---|---|---|---|
| Per capita tax rate | 1896 | 2120 | kcal/person/day |
| Total tax revenue | 6,413,300 | 9,553,360 | kcal/day |
| Population ($N$, $\hat{N}$ ) | 3382 | 4507 | individuals |
| Life expectancy ( $e_0, \hat{e}_0$ ) | 45 | 30 | years |
| Probability of survival to age 5 | 0.773 | 0.652 | |
| Food ratio ($E$, $\hat{E}$ ) | 1.820 | 0.668 | |
| Reproductive rate ($r$) | 0.0176 | 0.000 | |